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A DIGITAL COMPUTER APPROACH TO THE
UNSYMMETRIC RIGID BODY PROBLEM

Reference Systems Branch
Systems Avionics Division

March 1982

Final Report for Period January 1979 - April 1981

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The use of a computer approximation technique based on trial functions is investigated for the integration of the Euler equations of motion for a general rigid body rotating about a fixed point. The collocation method is evaluated against a Runge Kutta method for several cases consisting of various moments of inertia and applied torques. The collocation method includes two and four term polynomial trial functions which represent the angular velocities of the rigid body over intervals of time. The results of the methods

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are evaluated graphically against two known solutions and five other cases. The collocation method yields superior results to the Runge Kutta method.

FOREWORD

This report describes an in-house effort contracted by the author
in the Reference Systems Branch (AAAN), Systems Avionics Division (AAA),
Avionics Laboratory, Air Force Wright Aeronautical Laboratories (AFWAL),
Wright-Patterson Air Force Base, Ohio. This report is a thesis submitted
by Mr Richard Jacobs to the School of Engineering, University of Dayton,
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Master of Science in Engineering.

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LIST OF SYMBOLS

I_1, I_2, I_3	Principal moments of inertia
N_1, N_2, N_3	Torques about the body axes
ω_1, ω_2, ω_3	Angular velocities in body axes
ω_i	Angular velocity about an axis i
A	$(I_2 - I_3)/I_1$
B	$(I_3 - I_1)/I_2$
C	$(I_1 - I_2)/I_3$
M_1	N_1/I_1
M_2	N_2/I_2
M_3	N_3/I_3
F, G, H	Intermediate expression for computations of ω_1, ω_2, ω_3
c_1, c_2	
d_1, d_2, e_1, e_2	trial function coefficients
A_i	Coefficient $(I_j - I_k)/I_i$
M_i	Coefficient N_i/I_i

SECTION I

INTRODUCTION

The motion of a rigid body about a fixed axis is the next problem after the dynamics of particles in classical mechanics. The rigid body problem contains two parts, the first of which is the motion of the center of mass of the body, and the second of which is the motion of the body with respect to the center of mass. It is the second part which has remained an unresolved problem in classical dynamics. The problem can be completely specified mathematically, but in most cases the solution cannot be expressed in closed form.

The versatility of numerical methods along with the availability of digital computers has revolutionized the study of many problems. Indeed, the use of finite element methods has allowed the simulation of many difficult nonlinear problems in mechanics.

The following subsections contain a description of the general rigid body problem to be addressed in this research, as well as a brief historical section on the development of solutions to the problem. The objective and scope of the research is stated in subsection 3. The results of a literature search are found in subsection 4.

1. PROBLEM DESCRIPTION

Consider a rigid body of arbitrary shape, moving about a fixed point within the body due to the effect of applied torques or forces (see Figure 1). The angular momentum, L, of the body is given by

the expression

$$\bar{L} = I\bar{\omega} \tag{1}$$

where I = moment of inertia of the rigid body

$\bar{\omega}$ = angular velocity of the rigid body

and I and $\bar{\omega}$ are for convenience referred to the principal axes in the body.
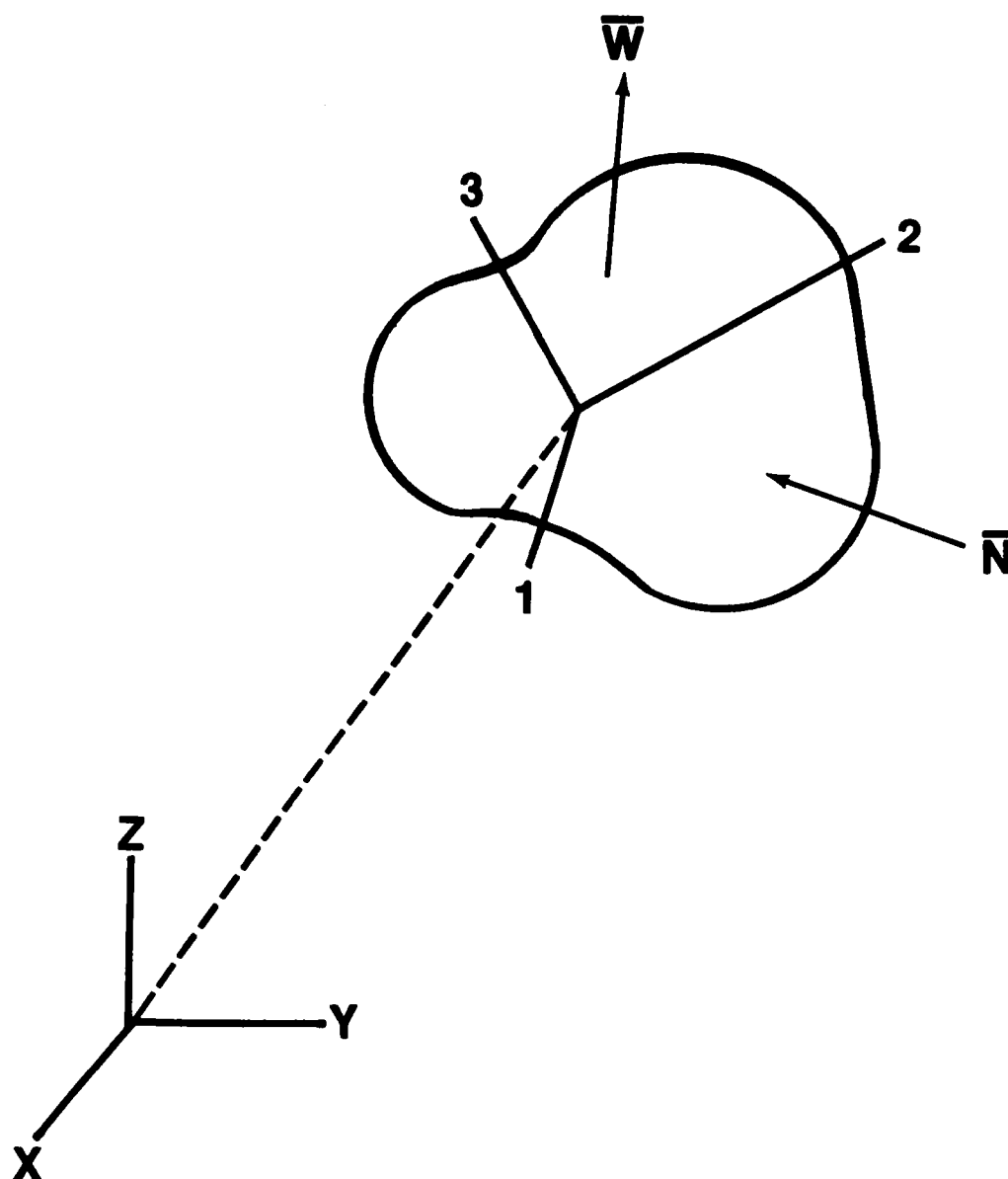
Figure 1.   Arbitrary Rigid Body

In an inertial frame, Newton's second law for rotational motion is written as follows

$$\frac{d\bar{L}}{dt} = \dot{\bar{L}} = \bar{N} \tag{2}$$

where $\bar{N}$ = vector of applied torques.

Equation 2 can be re-written for moving coordinates in the following form

$$\left(\frac{d\bar{L}}{dt}\right)_{space} = \left(\frac{d\bar{L}}{dt}\right)_{body} + \bar{\omega} \times \bar{L} \tag{3}$$

Substituting Equation 1 and Equation 2 into Equation 3, one arrives at

$$\bar{N} = I\dot{\bar{\omega}} + \bar{\omega} \times (I\bar{\omega}) \tag{4}$$

Equation 4 can be expanded for the three body axes

$$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \tag{5}$$

$$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 \tag{6}$$

$$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \tag{7}$$

Equations 5-7 are known as Euler's equations of motion for a rigid body rotating about a fixed point. The solutions ω_1, ω_2, ω_3 describe the motion of the rigid body in a set of coordinates fixed in or parallel to the body.

2. BACKGROUND

The rigid body problem of subsection 1 has well defined solutions when the body is symmetric, i.e., two or three moments of inertia are equal. However, when the rigid body is of arbitrary shape, i.e., unsymmetric, the solutions are either difficult to produce or unknown.

Classical dynamics texts such as Goldstein (Reference 1), MacMillan, (Reference 2), Whittaker (Reference 3), and Sommerfeld (Reference 4), give detailed accounts of special cases where solutions can be found. However, these solutions are in the form of elliptic integrals and other special functions, and require clever substitutions in order to manipulate the problem into a tractable one.

A concise summary of the historical development of solutions to the general rigid body problem can be found in Leimanis (Reference 5). The following names highlight the solution of the general rigid body problem.

1. Euler (1758): Symmetric body with no applied forces.
2. Lagrange (1788): Two moments of inertia equal and no applied forces.
3. Kovalevskaya (1888): Two moments of inertia equal, third moment of inertia equal to half the other two.
4. Staude (20th century): Unsymmetric top under force of gravity.
5. Bottema: Stability of Staude problem.
6. Grammel (1948): Use of approximating functions in a recursive scheme.

A brief description of Grammel's work can be found in Section III.

3. OBJECTIVE AND SCOPE

The problem outlined in subsection 1, as well as the known solutions in subsection 2, indicates the need for an approximation method based on modern digital computer methods. The need to study the motion of unsymmetric, or nearly symmetric rigid bodies might result from the design of gyroscopes, an unsymmetric mechanism, or a satellite antenna. Thus, the objective of this research is to investigate the use of an approximation method to obtain the description of motion of a general, unsymmetric rigid body under the influence of general torques or forces. The prevalence of finite element methods in other areas of mechanics provides a motivation to use a trial function technique analogous to finite element methods to approximate the motion of a rigid body.

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A secondary objective is to define a method which is inherently easy to use and not highly dependent on special knowledge or manipulations.

The scope of this research is limited to a comparison of weighted residual methods (discussed in Sections III and IV) with several well known numerical methods. *Thus the purpose of the work is concerned* with computer approximation methodology to generate numerical solutions to the general rigid body problem of subsection 1.

4. LITERATURE SEARCH

A literature search was conducted in order to identify any existing work in the area of approximate solutions in rigid body rotational dynamics. The University of Dayton file of theses was inspected, and no work had been done on this problem. A search was made of books on finite element methods. These sources included the references by Prenter (Reference 6), and Zienkiewicz (Reference 7), as well as the SAE proceedings (Reference 8). The material addressed under rigid body dynamics in finite element texts was generally concerned with the determination of the vibrational modes of rigid links, bars, or other members, and as such was not relevant to the issues in this research. A computerized search was done by way of the Lockheed data system at the Air Force Wright Aeronautical Laboratories (AFWAL) library, wherein the following files were examined:

> Compendex, Engineering Index
> Dissertation Abstracts
> ISMEC (Info. Service in Mechanical Engineering)
> NTIS (National Technical Information Service)
> DDC (Defense Documentation Service, government sponsored research)
> NASA

The results of the above searches were a series of computer printouts with finds, each containing titles, authors, locations, and abstracts. There were a few listings which led to further investigation, which are listed in the List of References and the Bibliography. For example, the papers by Likins (References 9-12), Kraige and Skaar (Reference 13), and

Huston and Passerello (References 14-16), were read for relevance as a result of the searches. In general, the listings referred to either multiple rigid body analysis (see Section III), or to finite element analysis of rigid plates, shells, or beams. No source was identified which duplicated the scope of the project.

SECTION II

A REVIEW OF METHODS OF DYNAMICS AND SOLVABILITY CONSIDERATIONS

The purpose of this chapter is to briefly review three widely used methods in dynamics and to show their similarities. These include the methods of Euler, Lagrange, and Hamilton. From these the problem in subsection 1 of Section I, is described and a choice of equations is made. In subsection 5 of this section a solvability consideration is described such that bounded, reasonable approximations can be expected.

Eulerian dynamics is concerned with the conservation of angular momentum, and can be thought of as another statement of Newton's second law for rotational dynamics. Lagrange's equations form an elegant alternative to the Euler-Newton method, making use of kinetic and potential energy statements and a function of generalized coordinates and velocities. In Hamilton's method, the n Lagrange, second-order, differential equations are replaced by 2n first-order, partial differential equations.

1. EULER'S EQUATIONS

Euler's equations for the rigid body were given in Section I by Equations 5-7, i.e.,

$$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3$$

$$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3$$

$$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2$$

These equations specify the angular velocity of the rigid body about its center of mass. The variables ω_1, ω_2, ω_3 are in body coordinates. The equations are coupled and nonlinear. They are coupled in that each variable ω_j occurs in equations other than the one containing its derivative $\dot{\omega}_j$. The equations are nonlinear in that each equation contains a product term $\omega_j\omega_k$. The solution of the equations is complicated by this nonlinear coupling.

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2. LAGRANGE'S EQUATIONS

In the formulation of Lagrange's equations, the crucial problems are finding the kinetic and potential energy expressions, solving the differential equations, and handling of any nonholonomic constraint forces. Lagrange's equations are given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q^j \tag{8}$$

where L = Lagrangian = T-V

T = kinetic energy

V = potential energy

q_j = generalized coordinates

$\dot{q}_j$ = generalized velocities

Q^j = generalized forces

The generalized coordinates q_j do not have to be the specific space coordinates of the problem, but can be "quasi-coordinates", i.e., coordinates suited to the problem at hand and not of necessity integrable combinations of the time derivatives of other coordinates. The quasi-coordinate concept is explained in Meirovitch (Reference 17, pages 137 and 157-160).

Lagrange's equations for quasi-coordinates for the rigid body problem are given by

$$\frac{d}{dt} [\frac{\partial T}{\partial \omega}] + [\omega] [\frac{\partial T}{\partial \omega}] = [N] \tag{9}$$

where [ω] is the vector of the coordinates ω_1, ω_2, ω_3 and [N] is the vector of generalized torques. The angular velocities ω_1, ω_2, ω_3 are not integrable time rates of change of angular displacements, but of the quasi-coordinates. The advantage in using this formulation is that the equations are written in terms of an orthogonal set of axes.

3. HAMILTON'S EQUATIONS

Hamilton's equations are more general than the methods of Euler and Lagrange. The Hamiltonian, H, is related to the Lagrangian, L, by the relationship

$$H = p_\alpha \dot{q}^\alpha - L \tag{10}$$

where p_α = generalized momentum components

$\dot{q}^\alpha$ = generalized velocities

Hamilton's equations are given by

$$\frac{dq^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} \tag{11}$$

$$\frac{dp^\alpha}{dt} = \frac{-\partial H}{\partial q^\alpha} \tag{12}$$

The H is expressed as functions of the coordinates q^α and momenta p^α, whereas in Lagrangian dynamics L is expressed in functions of the coordinates and velocities. Hamilton's equations for the rigid body problem can be found in Pars (Reference 18) and Webster (Reference 19). The resulting expressions for Equations 5-7 are six partial differential euqations involving the Euler angles.

Hamilton's equations do not facilitate the solution of particular problems, but lead to important theoretical generalizations in fields such as quantum, statistical, and celestial mechanics. There is no general technique for solution of the equations in closed form. However, the resulting first order differential equations are sometimes amenable to solution by well established methods in specific cases. In order to handle effectively a given problem, a knowledge of canonical or contact transformations, generating functions, and Jacobi's theory is required.

4. CHOICE OF METHOD

It can be shown that Euler's equations are equivalent to Lagrange's equations for quasi-coordinates (see Meirovitch, Reference 17, pages 157-169). The quasi-coordinate approach provides the most direct means

for representing a general rigid body of arbitrary shape in a set of orthogonal body coordinates. Hamilton's equations were judged to be too complicated to be appropriate and were not pursued any further.

For convenience, Equations 5-7 are re-written into a normalized form as follows:

$$\dot{\omega}_1 - A\omega_2\omega_3 = M_1 \tag{13}$$

$$\dot{\omega}_2 - B\omega_1\omega_3 = M_2 \tag{14}$$

$$\dot{\omega}_3 - C\omega_1\omega_2 = M_3 \tag{15}$$

where

$$M_1 = N_1/I_1, \quad A = (I_2 - I_2)/I_1$$

$$M_2 = N_2/I_2, \quad B = (I_3 - I_1)/I_2$$

$$M_3 = N_3/I_3, \quad C = (I_1 - I_2)/I_3$$

5. SOLVABILITY OF THE EULER EQUATIONS

In light of the non-linearity in Equations 13-15, it is fundamental to ask whether a solution exists and, if so, if it is unique. These equations are of first order, and there is much theory applicable to first order systems. See for example Birchoff and Rota (Reference 20, Chapter 6).

Each of the Euler equations can be re-written in the form

$$\dot{\omega}_1 = F(\omega_1, \omega_2, \omega_3, t) = M_1 + A\omega_2\omega_3 \tag{16}$$

$$\dot{\omega}_2 = G(\omega_1, \omega_2, \omega_3, t) = M_2 + B\omega_1\omega_3 \tag{17}$$

$$\dot{\omega}_3 = H(\omega_1, \omega_2, \omega_3, t) = M_3 + C\omega_1\omega_2 \tag{18}$$

The critical questions consist of the continuity of F, G, H and their partial derivatives, and the boundedness of F, G, H. If these conditions can be inferred, at least within reasonable intervals of computation, then the system has a unique solution. One means to test for boundedness is to introduce the Lipschitz condition, which states that for two points x and y on a region R:

$$|f(x, t) - f(y, t)| \leq L|x - y|, \ (x, t), (y, t) \ \epsilon \ R \tag{19}$$

L is the Lipschitz constant, which is an arbitrary scalar. Thus, if f is bounded, an L can be found which satisfies a Lipschitz condition on an interval. First order differential equation theory reveals that if $\dot{\omega}_1 = F(\omega_1, \omega_2, \omega_3, t)$ satisfies a Lipschitz condition on the domain t_1, t_2 then there is at most one solution $\omega_1(t)$ for a given initial condition, and thus uniqueness can be inferred. In the computer simulations in Section IV, this property is used to monitor the bound on consecutive approximations. An unbounded or divergent result is used to terminate the problem.

SECTION III

REVIEW OF COMPUTATIONAL METHODS

The objective of the project is to investigate the use of a computer approximation method for study of the general rigid body problem of Section I. In Section II, the choice of equations is made. The desirability of a trial function approach is a sub-objective, and because of it a number of finite element methods have been surveyed for relevance. The weighted residual method results as a trial function approach.

In studying the applicability of weighted residual methods, the need is also generated to compare the results with another known method. Subsection 1 contains a description of weighted residual techniques. In subsection 2, several known numerical methods are described. Subsection 3 provides a rationale for the choice of methods used in the project. In subsection 4, a brief review is provided of Grammel's trial function approach, as well as multi-rigid body analysis.

1. WEIGHTED RESIDUAL METHODS

A commonly used approach in approximation of ordinary differential equations is the weighted residual method. This method is characterized by the use of trial solutions with undetermined parameters, which are used to force a "residual" to approach zero. Four common weighted residual techniques are: collocation, subdomain, Galerkin, and least squares. The application of these techniques is illustrated in a simple and straightforward manner in Crandall (Reference 21).

In the weighted residual method a trial family of approximate solutions is selected. For example, consider the problem

$$\dot{x} = f(x, t), \quad x(0) = 1 \tag{20}$$

One could select the polynomial family

$$x = 1 + c_1 t + c_2 t^2 \tag{21}$$

as a trial function with undetermined parameters c_1, and c_2. The trial
function is chosen to satisfy the initial conditions of the problem
independently of the parameters c_1 and c_2. The residual, $R(t)$ is
formed by collecting all terms of the governing differential equation
together. In the example,

$$R = \dot{x} - f(x, t) \tag{22}$$

If the trial solution is the exact solution, then $R = 0$. The four tech-
niques are different criteria for determining the undetermined parameters
(c_1, c_2 in the example) over each interval.

a. Collocation

With the collocation method, distinct locations throughout the
interval are chosen at which the residual is set to zero. The number
of points corresponds to the number of undetermined parameters (two in
this case). The unknown parameters are found by solving a set of
simultaneous equations generated from the values of the residual at
the specified locations in the interval. The choice of locations
provides an effective weighting factor on the residual. It is possible
to design an algorithm to examine and adjust these weights for an optimum
between residual, computer time, etc. The interval size and trial
functions can also be adjusted according to the extent of non-linearity
in the problem. These characteristics can be used in the remaining
three methods.

b. Subdomain

In this method the interval is divided into as many subdomains
as there are parameters. The integral of the residual over each of the
intervals is set to zero to provide equations for determining the
unknown parameters. For example, using the subdomain (0, 1/2) and (1/2, 1),

$$\int_0^{\frac{1}{2}} R\, dt = 0 \tag{23}$$

$$\int_{\frac{1}{2}}^1 R\, dt = 0$$

c. Galerkin

In this case, weighted averages of the residual over the entire interval are required to vanish. *The weighting functions chosen are the same functions of t used in the trial family* (i.e., t and t^2, in the example). For example,

$$\int_0^1 tRdt = 0$$

$$\int_0^1 t^2Rdt = 0$$

(24)

d. Least Squares

In this case the criterion is to minimize the integral of the square of the residual over the interval. The formulas are given by

$$\frac{1}{2}\frac{\partial}{\partial c_1}\int_1^1 R^2dt = \int_0^1 R\frac{\partial R}{\partial c_1}dt = 0$$

$$\frac{1}{2}\frac{\partial}{\partial c_2}\int_0^1 R^2dt = \int_0^1 R\frac{\partial R}{\partial c_2}dt = 0$$

(25)

e. General Remarks

The four methods above can be summarized by the statement

$$\int_0^1 WRdt = 0$$

(26)

where W represents a weighting factor. The choice of weighting factors for the four methods discussed are: in collocation, the Dirac delta function $\delta(t)$; in subdomain, the unit step function; in Galerkin, the trial functions; and in least squares, the factors $\frac{\partial R}{\partial c_1}$, $\frac{\partial R}{\partial c_2}$. The

collocation method has an advantage in using the sifting property of the delta function in that the residual can be evaluated directly without integration, i.e.,

$$\int_{-\infty}^{\infty} R(t)\delta(t - t_1)dt = R(t_1) \tag{27}$$

Crandall shows an example problem with a known solution and compares the methods with a Taylor series approximation. In that case, all four give significantly better results than the Taylor series, with the least squares and subdomain methods the best over the interval chosen in the example. These results will vary depending on the problem, the interval size, and the trial functions.

2. NUMERICAL METHODS

The purpose of this section is to provide a brief summary of the numerical methods which are considered in this project.

a. Improved Euler Method

The basic Euler method consists of the construction of a tangent line approximation on a point by point basis along the curve $f(x, y)$. If $\dot{y} = f(x, y)$, with $y(x_0) = y_0$ then the Euler method consists of

$$y_{n+1} = y_n + hf(x, y) = y_n + h\dot{y}_n \tag{28}$$

where $n = 0, 1, 2, \ldots$
and h = step size.

The Euler method is very easy to use; however, the local formula error is proportional to h^2, and thus the method is not very accurate.

In the improved Euler method, the above formula is complemented by the average of the values at two points, thus improving the tangent line estimate. The improved Euler formula is given by

$$y_{n+1} = y_n + \frac{\dot{y}_n + f[x_n + h, y_n + h\dot{y}_n] h}{2} \tag{29}$$

The error is proportional to h^3. This is the simplest "predict-correct" method.

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b. Three Term Taylor Series

The Euler formula (Equation 28) is in essence a two term Taylor series approximation. This can be made more accurate by using the first three terms. Thus,

$$y_{n+1} = y_n + h\dot{y}_n + \frac{h^2}{2} \ddot{y}_n \qquad (30)$$

where $\dot{y}_n = f(x, y)$

and $\ddot{y}_n = f_x(x_n, y_n) + f_y(x_n, y_n)\dot{y}_n$

The local formula error is proportional to h^3 as in the improved Euler method. The three term Taylor series method requires the calculation of partial derivatives f_x, f_y. In principle, even higher term series can be used, but the difficulty in computing the terms with the higher derivatives makes the method very awkward to use.

c. Runge Kutta Method

The Runge Kutta method involves a weighted average of values of $f(x, y)$ taken at different points in the interval $x_n \le x \le x_n + 1$. A commonly used Runge Kutta technique is characterized by

$$y_{n+1} = \frac{h}{6} [kn_1 + 2kn_2 + 2kn_3 + kn_4] \qquad (31)$$

where

$$kn_1 = f(x_n, y_n)$$
$$kn_2 = f(x + \frac{h}{2}, y_n + \frac{h}{2} kn_1)$$
$$kn_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2} kn_2)$$
$$kn_4 = f(x_n + h, y_n + h\, kn_3)$$

The local formula error is proportional to h^5, and thus a more accurate formula is obtained at the cost of more computation. The above formula is equivalent to a five term Taylor series approximation; the "k" factors substitute for the need to compute partial derivatives. This method is

one of the most widely used methods for the numerical integration of
ordinary differential equations in engineering analysis, and is one given
attention with the collocation method in this project.

3. SUMMARY AND CHOICE OF METHODS

The Runge Kutta method is selected over the Euler and Taylor methods
because of the improved formula error and ease of use over series cal-
culations. The weighted residual methods are chosen because they provide
a means to approximate the velocities in the Euler equations by a polynomial,
trial function process. This will be explained in more detail in
Section IV.

4. GRAMMEL'S WORK AND MULTI RIGID ANALYSIS

This subsection briefly reviews the approximation techniques
attempted by Grammel and the computer methods known on multiple rigid body
analysis. These methods have been identified in the literature search
stage of the project. They are not directly relevant to the problem under
study, but their prevalence warrants the following summary.

a. Grammel's Work

The work of Richard Grammel (References 22-24) is summarized here
because it represents a twentieth century attempt at the solution of the
unsymmetric top problem by means of an iterative, trial function process.

Grammel's work is of a more analytic than computation nature.
Grammel considers Equations 5-7 and studies trial solutions of the form

$$\omega_1 = a + \epsilon_1 e^{Qt}$$
$$\omega_2 = b + \epsilon_2 e^{Qt}$$
$$\omega_3 = c + \epsilon_3 e^{Qt} \tag{32}$$

where Q is a complex variable.

Grammel is able to establish stability criteria and to in some cases
solve the problem. The approach does not make use of the weighted
residual method, and the mathematical expressions in the higher iterations

become cumbersome. However, foundations are laid prior to the introduction of large scale digital computers.

b. Multi-Rigid Body Analysis

Multiple Rigid Body analysis is an efficient, computer-oriented program to study and resolve the motion of complex systems of linked rigid bodies. As such, it is not relevant to the rigid body problem under study here. It would be applicable to, for example, a satellite with flexible appendages, an orbiting antenna composed of connected rigid links, or the bones of the human spine, rib cage, neck, etc. A typical chain system is shown in Figure 2. Finite segment and multi-rigid body modeling programs contain an efficient means to label each link and its connection, then to reduce the system to its equivalent motion using algebraic operations and D'Alembert's form of Lagrange's equations. These methods are in essence a computerized application of planar mechanism theory. As such they do not address the issues of interest in the rotation of unsymmetric tops. However, two useful concepts are used in these formulations. One is the use of hybrid or quasi-coordinates, which is discussed in Section II under Lagrangian dynamics. Second is the use of Euler parameters or quaternions to eliminate singularities in the model.
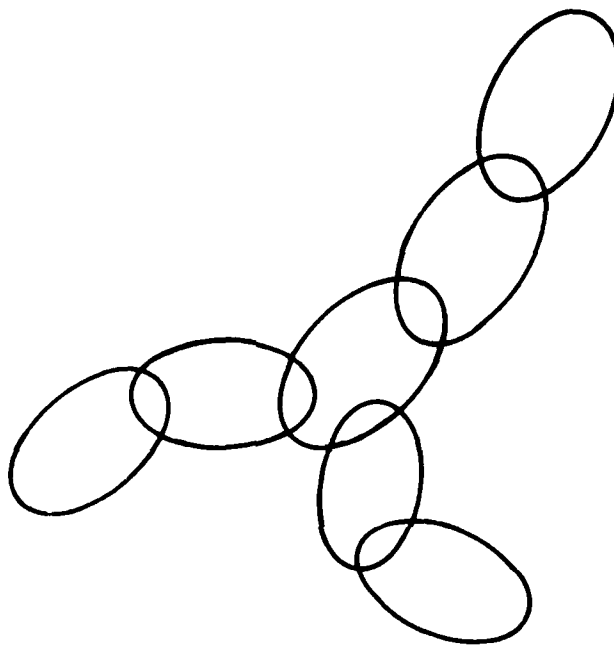
Figure 2. Multiple Rigid Body

SECTION IV

COMPUTER SIMULATION OF THE RIGID BODY

This chapter describes the application of the collocation method
to the Euler Lagrange equations (13-15) in a trial function scheme.
In subsection 1, the rationale for trial functions is explained. In
subsection 2, the algorithm for use of collocation is detailed. In sub-
section 3, the algorithm for a four term Runge Kutta method is outlined.

The capability to model problems with time varying torques is added
to both the Runge Kutta and collocation programs. The idea to include this
capability results from a survey of Grammel's papers. Examples of such
torques acting on a top result from a rotating electromagnetic field, a
change in forces acting on a satellite, or the influence of a small retro
jet on a space body when the percent change of mass is insignificant.

1. TRIAL FUNCTION APPROACH

The Euler Lagrange equations (13-15) can be summarized in the
following form

$$\dot{\omega}_i - A_i \omega_j \omega_k = M_i \tag{33}$$

where i, j, and k range from 1 to 3 in cyclic order. The nonlinearity
in these equations is caused by the coupling between the $\omega_j \omega_k$ terms.
If the rigid body were symmetric, e.g., sphere, all such terms would be
zero and the problem would reduce to the simple integration of $\dot{\omega}_i = M_i$
with solution $\omega_i = M_i \Delta t$. A nearly symmetric top would have coefficients
A_i nearly zero and the equations might be modeled by first considering
the uncoupled solution

$$\omega_i = M_i \Delta t$$

to obtain an initial estimate of the ω_i terms, then re-adjusting them
to fit the equations with the coupled terms $\omega_j \omega_k$ formed from the initial
estimates. Thus a linear estimate $M_i \Delta t$ could be used for each interval.
The problem with this approach is the limitation on a velocity term ω_i
which is in reality changing throughout the interval Δt. In other words,

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the linear estimate $M_i \Delta t$ allows only constants for ω_i. A more general top would require that the intervals Δt be made small enough that the constant ω_i terms still track the solution. In practice the interval will reach a lower limit and the approximation will fail to be effective.

The choice of trial functions is based on the level of dynamical behavior expected in the problem, and by the computational or algorithmic constraints on the programmer. A linear top problem would only require linear trial functions of the form $c_1 t$ ($c_1 = M_1$). A general unsymmetric problem would require higher order trial functions. In this effort the choice is made of two and four term trial functions of the polynomial family. The polynomial allows the derivative terms $\dot{\omega}_i$ to be easily expressed. A four term trial function allows $\dot{\omega}_i$ to be easily expressed. A four term trial function allows $\ddot{\omega}_i$, an acceleration term, to vary up to

$$c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3$$

and is thus a rather general motion. The cost in programming is the expansion from the solution of two equations in two unknown parameters in each interval to the solution of four equations in four unknown parameters. Higher order trial functions can be written, but the improvements are limited as the higher powers of t add only small contributions to the estimation.

2. COLLOCATION PROGRAM

If the above is generalized to allow at least a first order change in ω_i throughout the interval then the idea of trial functions follows. Consider an initial estimate of ω_i given by

$$\omega_i = \omega_i(0) + c_1 t + c_2 t^2 \tag{34}$$

The $\omega_i(0)$ satisfies the initial condition on ω_i. The c_1, and c_2 are parameters to be determined in the interval Δt. Proceeding with an initial estimate of ω_1, ω_2, ω_3 the ω_j ω_k terms are calculated and used to solve the equations for the interval. The reason for separation of the

problem into a first estimate and a correction is the non-linear coupling of the Euler equations, and the fact that $\omega_j \, \omega_k$ are not available to calculate ω_i, etc.

To illustrate the above procedure, let

$$\omega_1(1) = \omega_1(0) + c_1 t + c_2 t^2$$
$$\omega_2(1) = \omega_2(0) + d_1 t + d_2 t^2 \qquad \qquad (34a)$$
$$\omega_3(1) = \omega_3(0) + e_1 t + e_2 t^2$$

This allows $\dot{\omega}_1 = c_1 + 2c_2 t = $ a linear function of time over the interval under study. Now form

$$\dot{\omega}_1 = F = M_1 + A\omega_2 \omega_3$$

where $\omega_2 \, \omega_3$ are the initial estimate and are found from the initial conditions $\omega_2(0)$ and $\omega_3(0)$. Upon substitution of the trial functions for $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$, the following expressions result

$$c_1 + 2c_2 t = F = M_1 + A\omega_2(0)\omega_3(0)$$
$$d_1 + 2d_2 t = G = M_2 + B\omega_1(0)\omega_3(0)) \qquad \qquad (35)$$
$$e_1 + 2e_2 t = H = M_3 + C\omega_1(0)\omega_2(0)$$

For the initial estimate $\omega_1(1)$, $\omega_2(1)$, $\omega_3(1)$ one can solve each of the above three equations separately by expanding each by one of the four weighted residual methods discussed in subsection 1 of Section III. Using collocation, two points within the interval are selected and each of the three equations is evaluated at the two points to determine the coefficients c_1, c_2, ... e_2. See Figure 3 for a typical interval. For example

$$c_1 + 2c_2 t_1 = F(t_1)$$
$$c_1 + 2c_2 t_2 = F(t_2) \qquad \qquad (36)$$
$$t_1, \; t_2 \epsilon (t_0, \; t_3)$$

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$$T_0 \qquad T_1 \qquad T_2 \qquad T_3$$

Figure 3. Time Increment for Collocation

For the initial estimate it is assumed that F is constant over the interval, since only the initial values of ω_2, ω_3 are available. The above can be rewritten in matrix form as follows

$$\begin{bmatrix} 1 & 2t_1 \\ 1 & 2t_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad (36a)$$

Having determined $c_1 \ldots e_2$, initial estimates of ω_1, ω_2, ω_3 are constructed *from the* constitutive definitions (Equation 34a). These estimates are used to now calculate the $\omega_j \omega_k$ terms in F, G, H and a corrected calculation can be made which allows these terms to vary with time over the interval. When the corrected ω_i has been calculated, the process iterates forward with the values of $\omega_1(1)$, $\omega_2(1)$, $\omega_3(1)$ as the new initial conditions.

The subdomain, Galerkin, and least squares criteria all require that an integration be performed on each interval of time, whereas the collocation method makes use of the sifting property (Equation 27) to evaluate the residual equations at specific points in each interval. The integration requires that the terms of the $\omega_j \omega_k$ be written out and and evaluated over each interval. For a two term model, one such product looks like

$$(c_1 t + c_2 t^2) (d_1 t + d_2 t^2)$$

In the collocation method, one need only to calculate each ω_1, ω_2 and ω_3 term at the points t_1, t_2 within the interval (see Figure 3) and to apply

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them in the ω_j ω_k terms, i.e., long expressions of trial functions multiplied together are not needed.

The subdomain, Galerkin, and least squares methods were implemented in Fortran programs, but were later dismissed from further consideration because they failed to produce meaningful results for the test cases where the solution is known. It is expected that the computer error generated from the long expressions of ω_j ω_k terms caused this error.

The program flow for the collocation method is shown in Figure 4. The locations used in the two term models are 1/3 and 2/3 over each interval; for the four term model, the locations are .2, .4, .6, and .8. The specific algorithm for one interval procedes as follows:

    a. DT = .01

    b. T1 = 1/3DT, T2 = 2/3DT

    c. Solve for c_1, c_2

$$\begin{bmatrix} 1 & 2t_1 \\ 1 & 2t_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{matrix} M_1 + A\omega_2(0)\omega_3(0) \\ M_1 + A\omega_2(0)\omega_3(0) \end{matrix}$$

and likewise in d_1, d_2, e_1, e_2.

    d. Form

$$\omega_1(1) = \omega_1(0) + c_1 t + c_2 t^2$$
$$\omega_2(1) = \omega_2(0) + d_1 t + d_2 t^2$$
$$\omega_3(1) = \omega_3(0) + e_1 t + e_2 t^2$$

    e. Now calculate

$$F_1 = F(t_1) = M_1 + A\omega_2(t_1)\omega_3(t_1)$$
$$F_2 = F(t_2) = M_1 + A\omega_2(t_2)\omega_3(t_2)$$

and likewise in remaining variables.
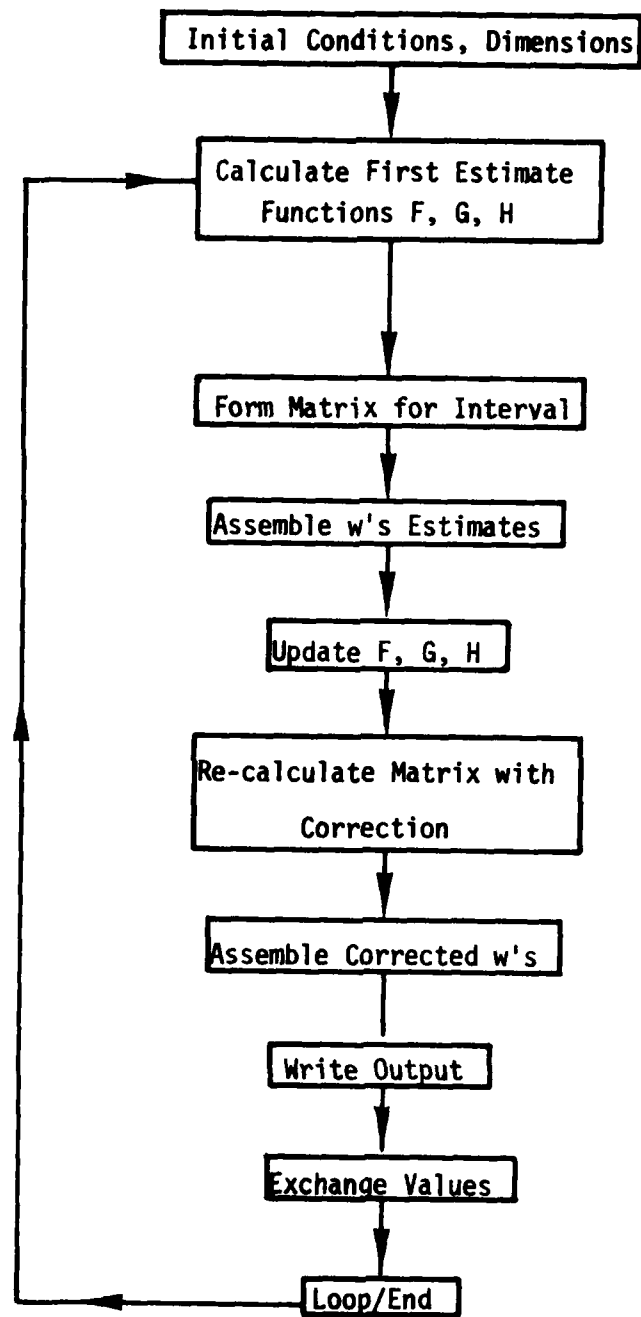
Figure 4. Collocation Program Flow

f. Solve

$$\begin{bmatrix} 1 & 2t_1 \\ 1 & 2t_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} F(t_1) \\ F(t_2) \end{bmatrix}$$

g. Assemble

$$\omega_1(1) = \omega_1(0) + c_1 t + c_2 t^2$$

$$\omega_2(1) = \omega_2(2) + d_1 t + d_2 t^2$$

$$\omega_3(1) = \omega_3(0) + e_1 t + e_2 t^2$$

h. Print results for interval

i. Exchange $\omega_i(1)$, $\omega_i(0)$ and iterate to next interval.

3. RUNGE KUTTA PROGRAM

The Runge Kutta method from Section III, Equation 31 is implemented in a Fortran program for comparison with collocation. The use of Equation 31 is straightforward; the Runge Kutta "k" terms are calculated in each interval and the estimates for ω_1, ω_2, ω_3 are assembled from these terms. The interval size for both the Runge Kutta and collocation is chosen to be .01. In comparing the Runge Kutta and collocation methods, the same step size is used throughout. A copy of the Runge Kutta program can be found in the appendix. The program flow is illustrated in Figure 5.

The Lipschitz condition is used in the Runge Kutta program in order to guard against unbounded approximations. An initial Lipschitz constant of 500 is used; in some cases, larger values are used.
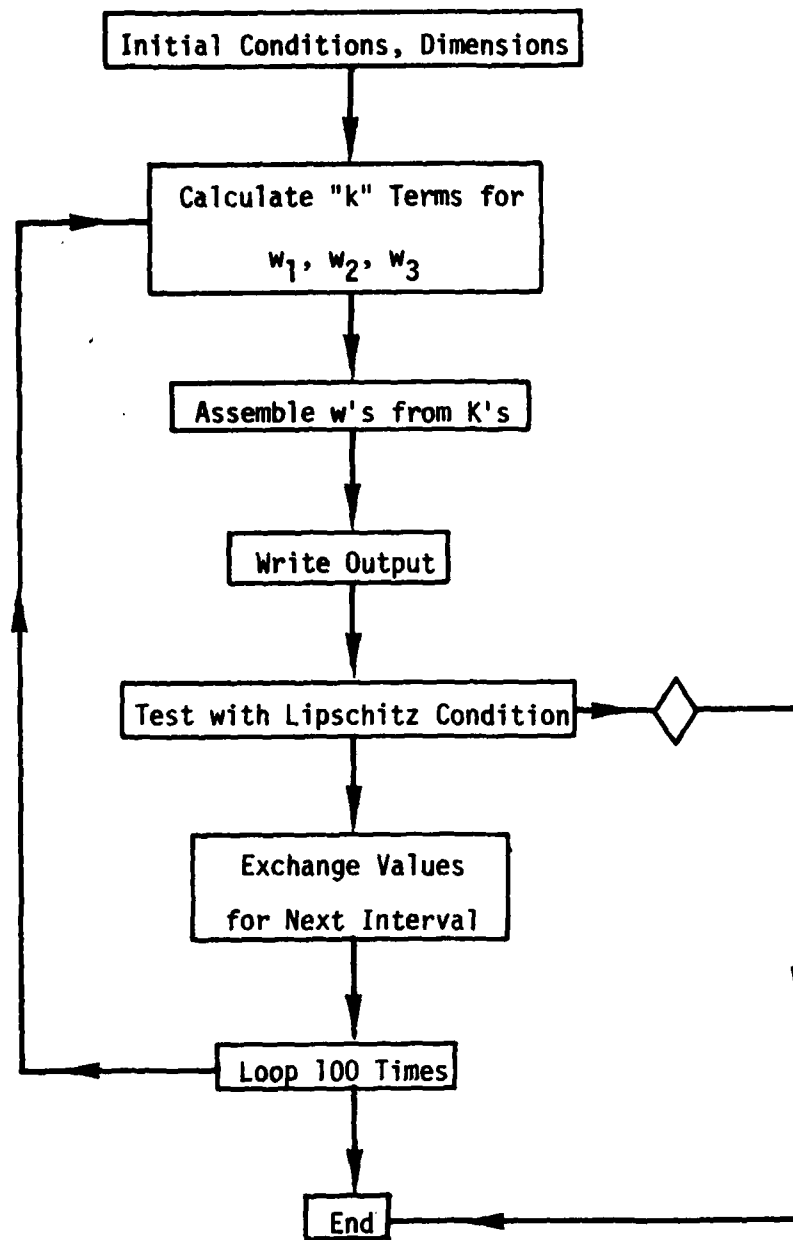
Figure 5.   Runge Kutta Program Flow

SECTION V

RESULTS

The collocation programs, as well as the Runge Kutta program from Section IV, are evaluated for effectiveness by using a number of test cases. The test cases represent rigid bodies of given moments of inertia and torques. Subsection 1 of Section V details the test cases used in this work. In subsection 2, a discussion of the graphical results is included, as well as a brief discussion of error. In subsection 3, a brief discussion is provided on computer execution time, complexity, and general concerns.

1. TEST CASES

The following test cases are used in evaluation of computer approaches from Section IV. (See Table 1.)

Case 1 is a simple, linear, uncoupled top, and is included for validation of the methods. The solutions are constant functions of time.

Case 2 is a force free, symmetric top, and can be found in Goldstein (Reference 1, pages 161-162). The solution is known and is given by

$$\omega_1 = \sin 5t$$

$$\omega_2 = \cos 5t$$

$$\omega_3 = 10$$

Case 3 is a small perturbation of the problem in case 2.

Case 4 is an unsymmetric top with no torques.

Case 5 is the same top with unequal torques added.

In Case 6 a grossly unsymmetric top is modeled.

In Case 7 the top of Cases 4 and 5 is now given time varying torques.

The values for the moments of inertia and torques are chosen to produce reasonable numbers on output and have no special physical significance.

In Cases 1 and 2 an objective study of the error can be made against known solutions. In the remaining cases, the solutions are not known and one can only compare the results qualitatively.

TABLE 1

TEST CASES

1. $I_1 = I_2 = I_3 = 40$
   $N_1 = N_2 = N_2 = 100$

2. $I_1 = I_2 = 40, \quad I_3 = 20$
   $N_1 = N_2 = N_3 = 0$

3. $I_1 = 41, I_2 = 39, I_3 = 20$
   $N_1 = N_2 = N_3 = 0$

4. $I_1 = 40, I_2 = 30, I_3 = 20$
   $N_1 = N_2 = N_3 = 0$

5. $I_1 = 40, I_2 = 30, I_3 = 20$
   $N_1 = 100, N_2 = 200, N_3 = 150$

6. $I_1 = 200, I_2 = 10, I_3 = 20$
   $N_1 = 100, N_2 = 200, N_3 = 150$

7. $I_1 = 40, I_2 = 30, I_3 = 20$
   $N_1 = 100, N_2 = 200, N_3 = 150$
   (initially)
   $N_1 = \sin 100t, N_2 = \cos 100t, N_3 = t$
   (as $t > 0$)

2.   GRAPHICAL RESULTS

The results of the seven test cases are plotted in Figures 6 through 12. The following is a discussion of these results.

In Case 1, Figure 6, all three programs (2 and 4 term collocation, Runge Kutta) show good correlation and coincide with the known solution. There was no deviation in the results of any of the programs from the known solution.

In Case 2, Figure 7, the known solution is plotted along with the 2 and 4 term collocation and Runge Kutta programs. The collocation programs show a good correlation with the known solution and are coincident with the known solution for the graph shown. The error was found to be of the order of .05 to 1% and constant. The Runge Kutta program gave poor results, and terminated after only 25 steps due to the Lipschitz condition. The ω_1 curve can be seen to be growing beyond the expected sine solution. The Runge Kutta program has been rechecked for an error, but none has been found.

In Case 3, Figure 8, the solution of Case 2 is slightly perturbed such that ω_3 will no longer be a constant. The collocation programs concur with one another. The Runge Kutta program is again diverging.

In Case 4, Figure 9, an unsymmetric top is modeled. The collocation programs concur, but the Runge Kutta program fails.

In Case 5, Figure 10, one notices some differences between the 2 and 4 term collocation models. The 4 term model is assumed to be the more reliable result because of the extra terms and the experiment described in Case 6. Figure 10 is divided into an A and B part to illustrate the different curves.

In Case 6, Figure 11, one sees major differences between the models. The 2 term model diverged and is not shown. The 4 term model illustrates very dynamic results. Because of the disparity between the 2 and 4 term results, it is judged that the motion being simulated is too fast for the 2 term model. To check the 4 term collocation model, 3 and 5 term models have been used. The results of the 3, 4, and 5 term collocation models concur.

In Case 7, Figure 12, the 2 and 4 term models concurred fairly well.

3. SUMMARY COMMENTS

The collocation programs use approximately twice the execution times of the Runge Kutta program to compute 100 steps. However, this is of no special concern as the average execution times are of the order of .3 seconds, which is negligible by current computing standards.

The collocation programs are of about the same level of complexity to program and use as the Runge Kutta method. The collocation methods require an understanding of the use of a matrix subroutine.

The step size has been changed from .01 to .001 for several cases in order to evaluate error trends in the programs. The collocation programs provide generally better results with this step size (with the exception of Case 6, 2 term) whereas the Runge Kutta model does not improve.
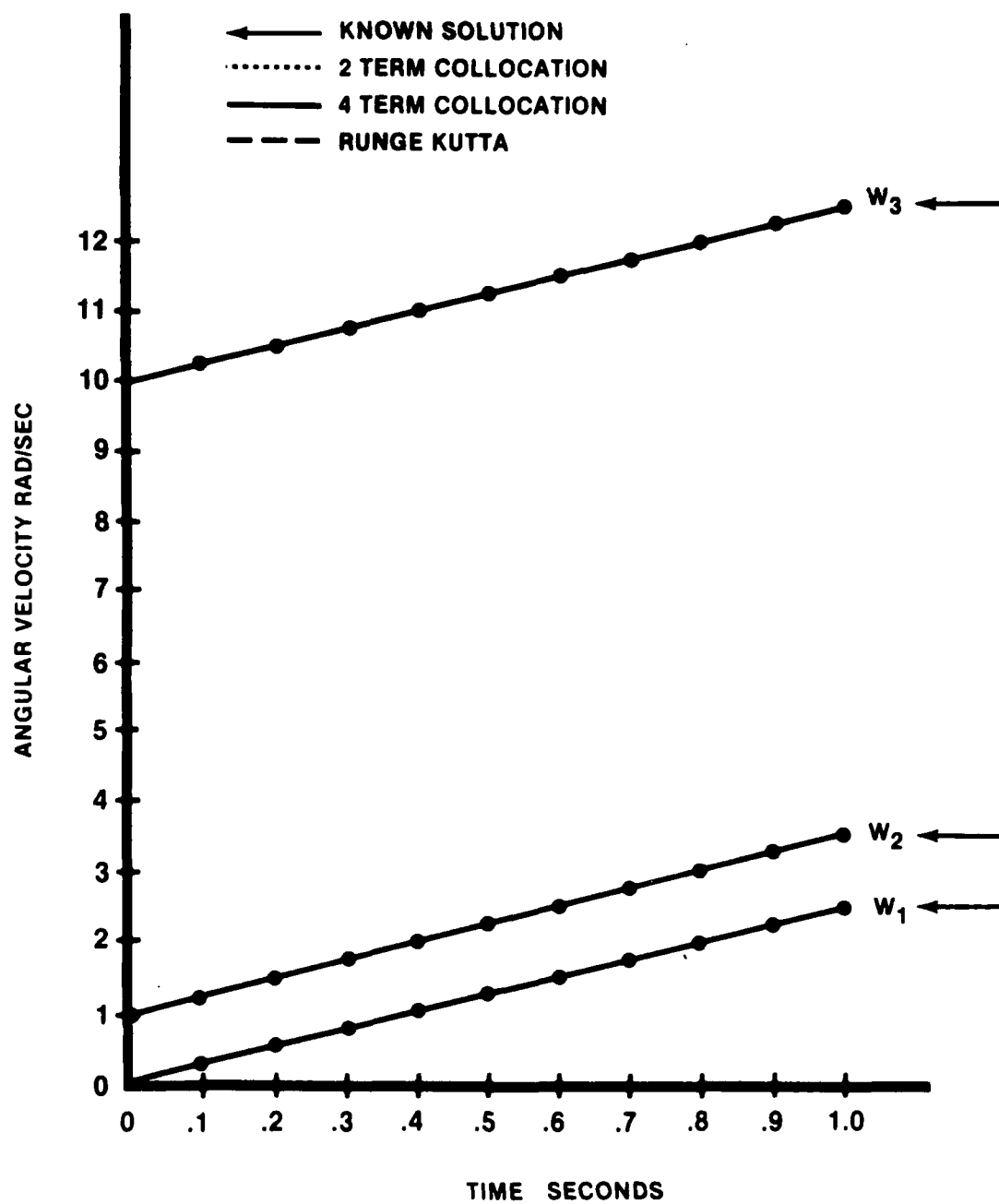
Figure 6. Case 1
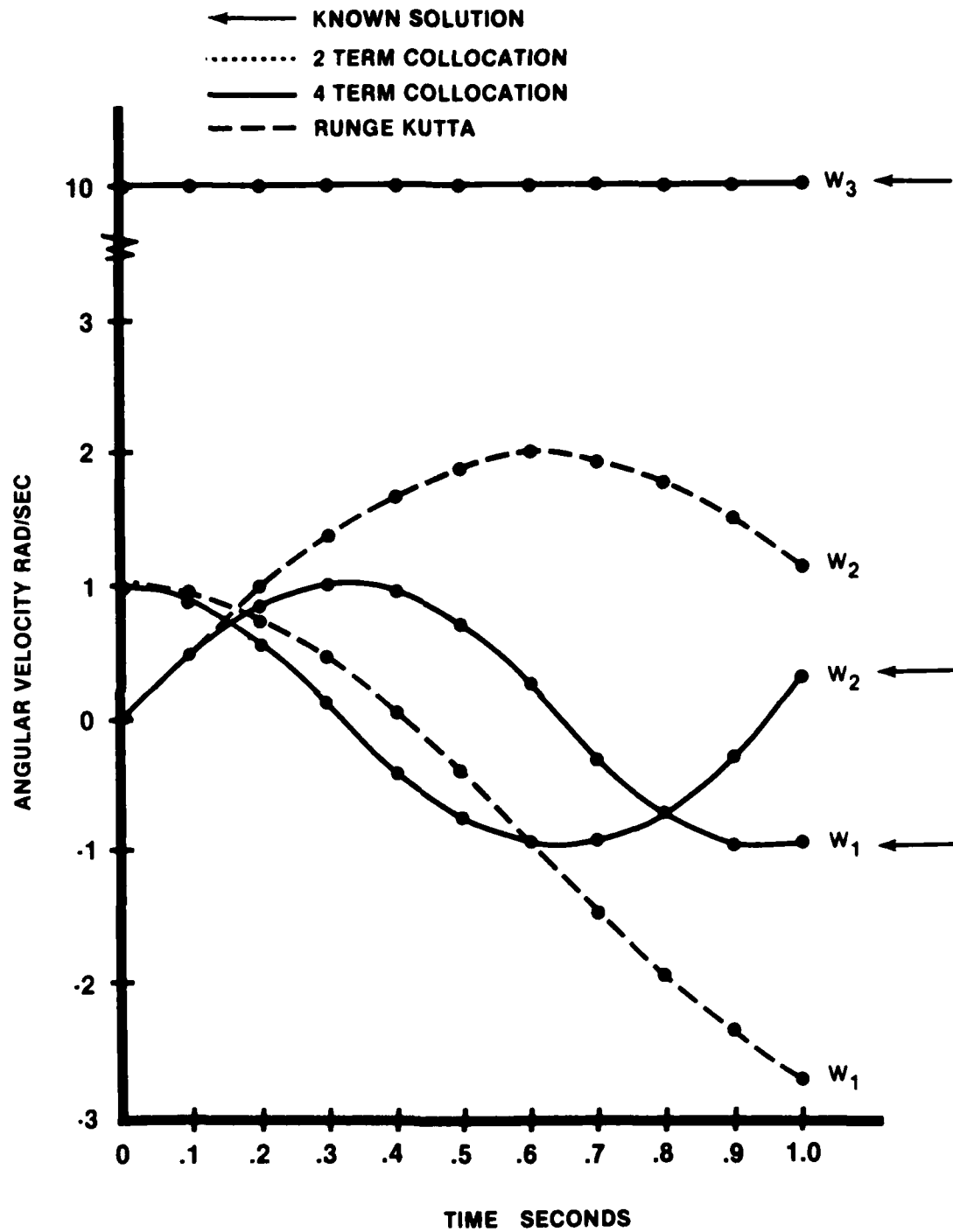
Figure 7. Case 2
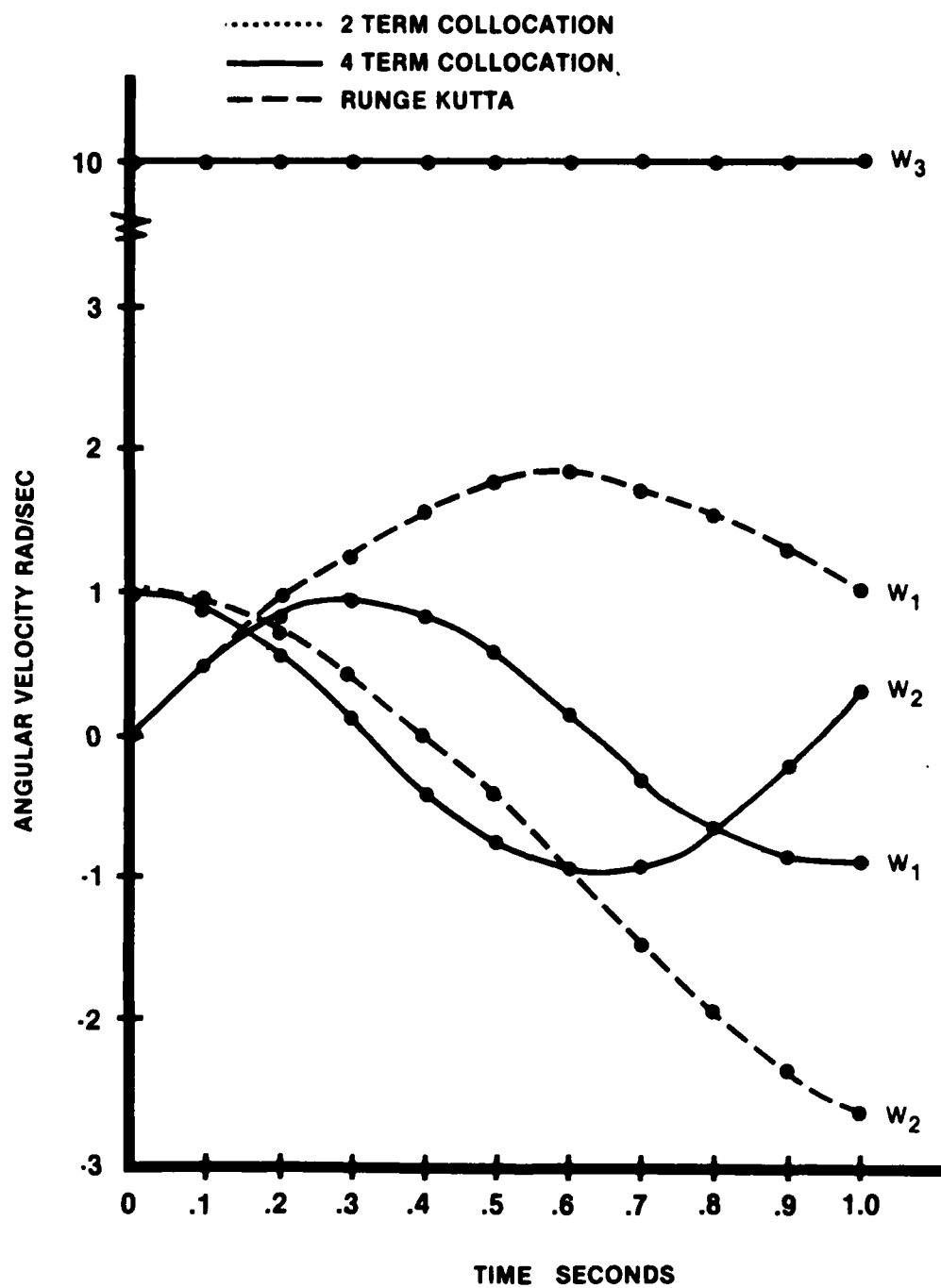
Figure 8.   Case 3
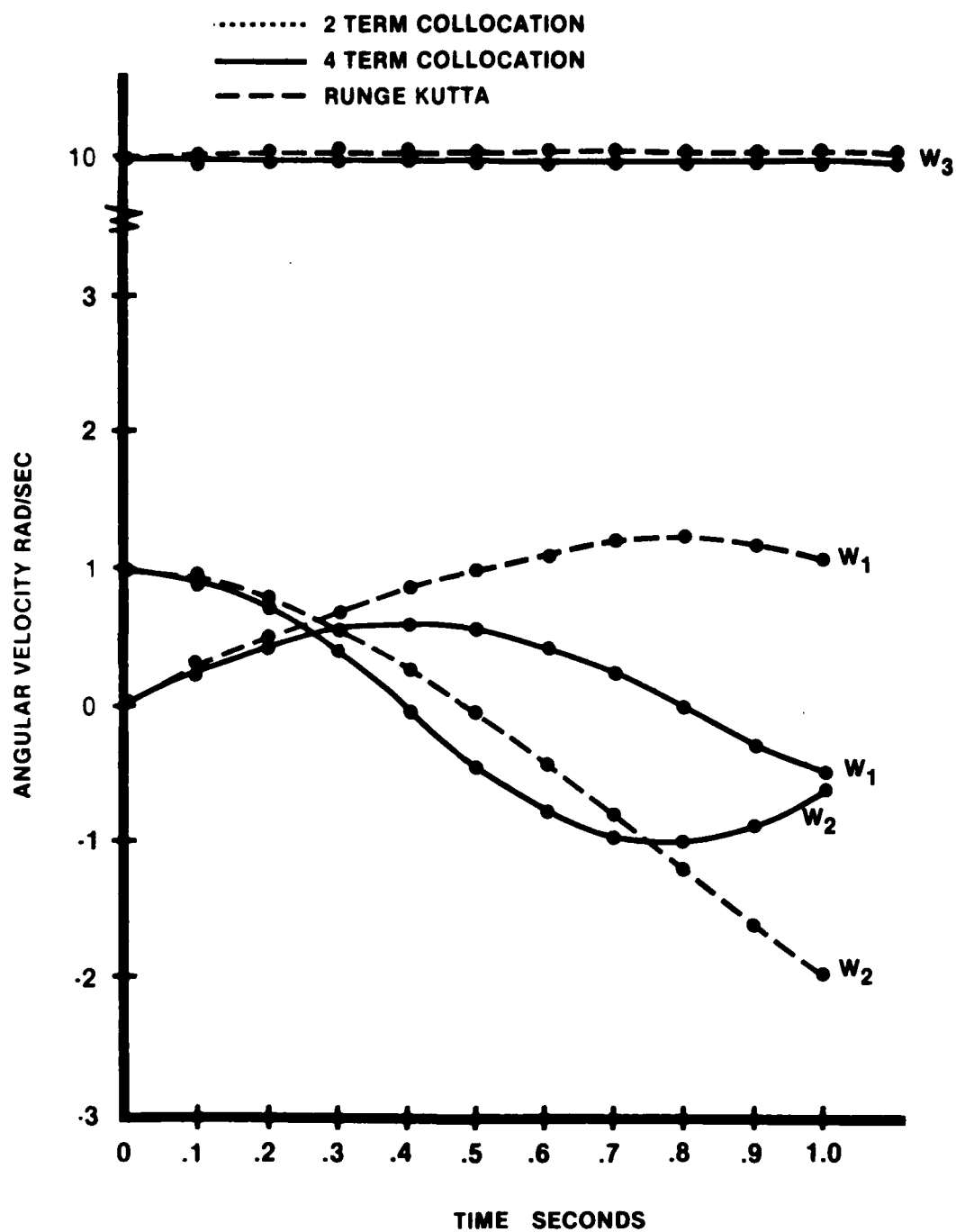
Figure 9. Case 4
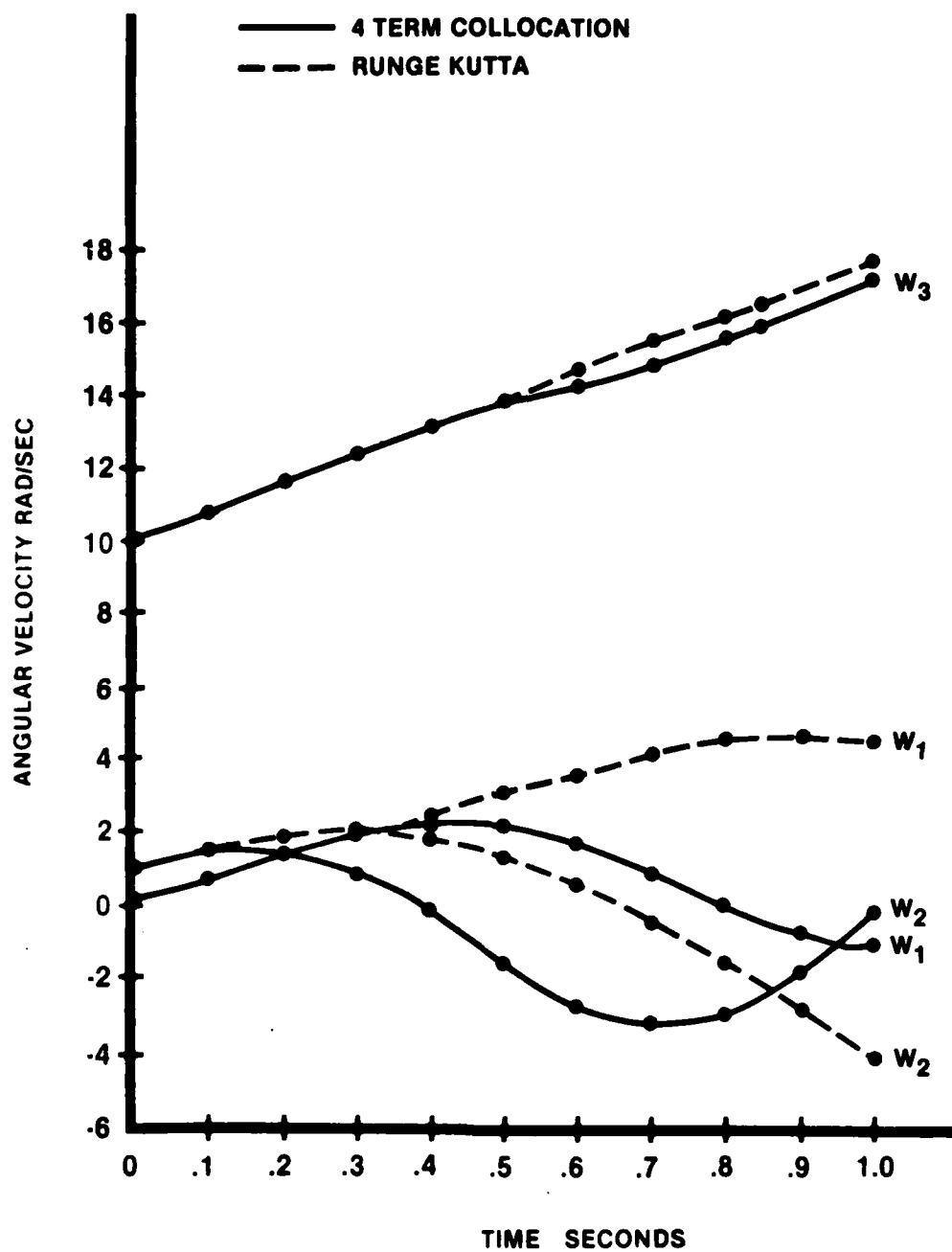
Figure 10. Case 5
PART A
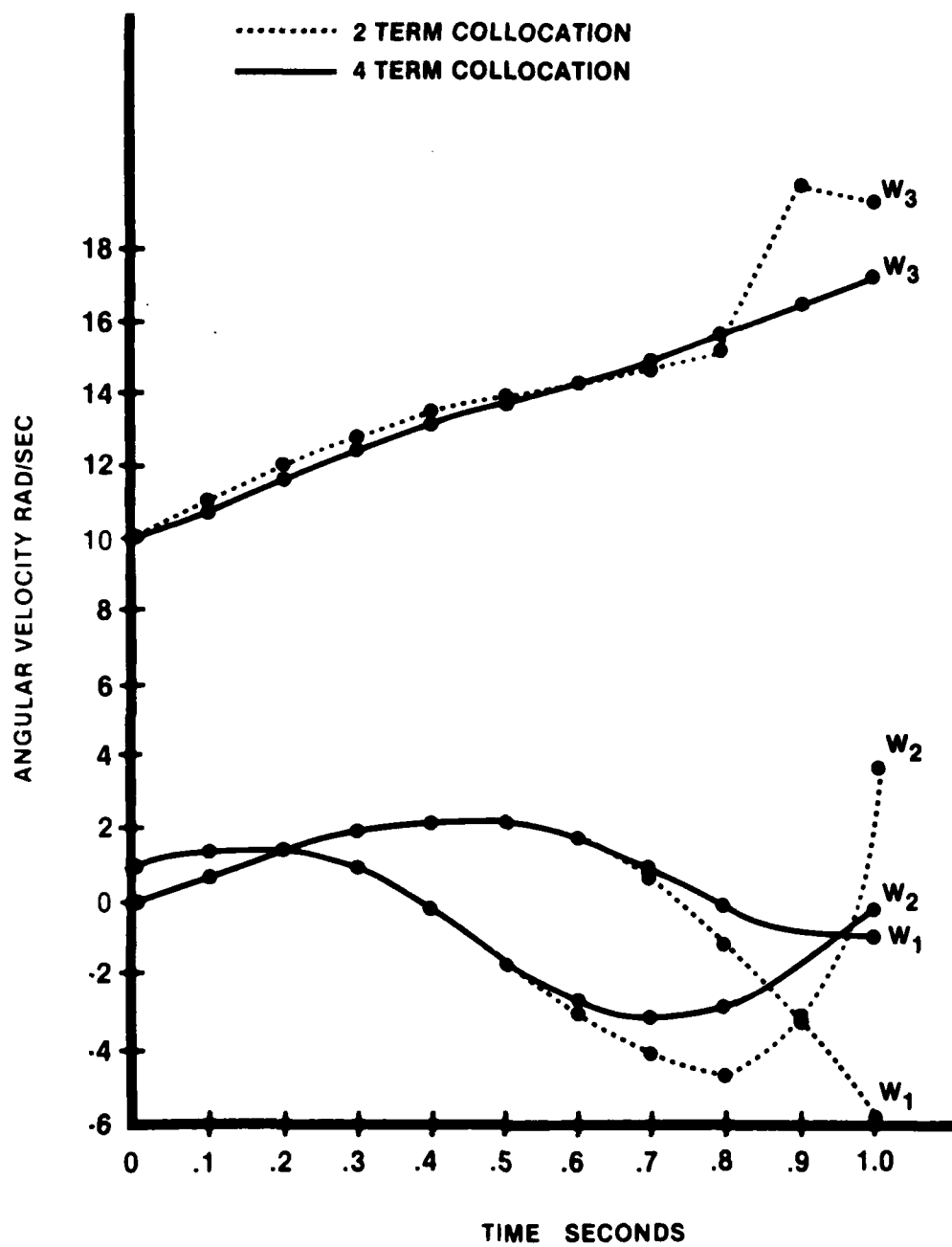
Figure 10. (Concluded)

PART B
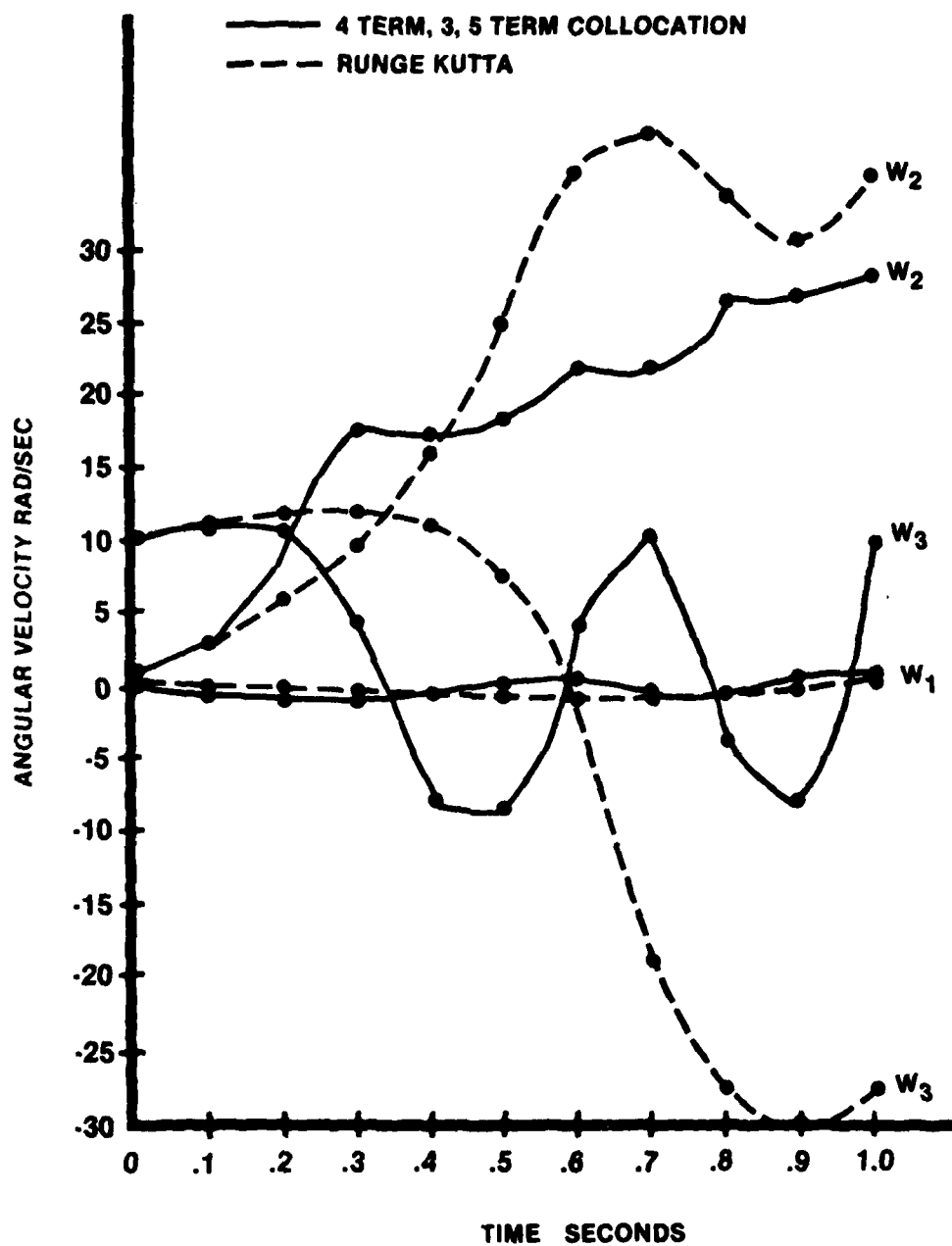
Figure 11.   Case 6
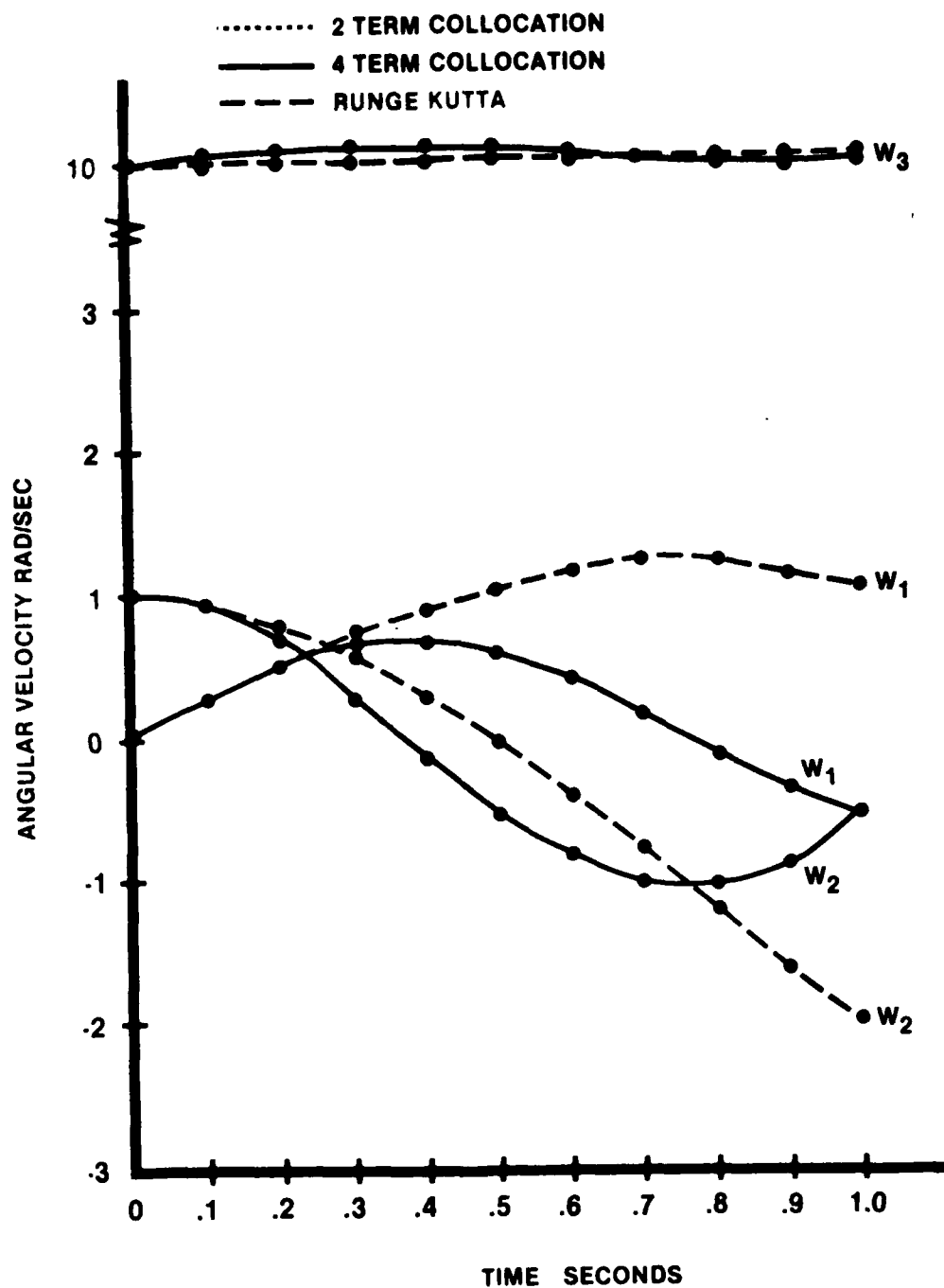
Figure 12.   Case 7

SECTION VI

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The general rigid body problem of subsection 1 of Section I has been modeled by a trial function technique on a digital computer. The trial functions represent estimates of angular velocities of the rigid body over intervals of time. The results of the computer models have been studied against known solutions and graphically against another numerical method.

The results of Section V allow a number of conclusions to be made. The weighted residual method is a successful trial function approach to the rigid body problem. In particular, collocation is the successful technique of the weighted residual methods. Collocation is superior to the Runge-Kutta method employed in this project. Collocation is of about the same level of computational complexity to program as Runge-Kutta. The results indicate that a three or four term collocation model is probably sufficient for most problems.

The above conclusions leave a number of issues open for further investigation. The following recommendations are suggested for further work:

1. Experiment with alternative step sizes and trial functions for collocation, and study practical matters of convergence and error.

2. Extend the collocation method to a predict-correct technique and evaluate against prior results.

3. Compare the results of this project with another well established numerical integration routine, e.g., Adams predictor corrector.

4. Use the results from above to investigate classical solutions in the literature.

5. Develop a means for transforming ω_1, ω_2, ω_3 from body coordinates to space coordinates and integrate to Euler angles.

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APPENDIX AND PROGRAMS

2 Term Collocation

4 Term Collocation

Runge Kutta

```
L,A
 100-       PROGRAM FEM1(INPUT,OUTPUT,TAPE5,TAPE6)
 110-       DIMENSION A(2,2),B(2,6),USA(12)
 120-       REAL I1,I2,I3,N1,N2,N3,M1,M2,M3,MS1,MS2,MS3
 130-       REAL LIP
 140-       DIMENSION U1(6),U2(6),U3(6),V1(6),V2(6),V3(6)
 150-       TIME=0.
 160-C
 170-C      COLLOCATION PROGRAM
 180-C      TWO TERM APPROXIMATION
 190-C      MOMENTS OF INERTIA
 200-       I1=40.
 210-       I2=40.

 220-       I3=20.
 230-C
 240-C      TORQUES
 250-       N1=0.
 260-       N2=0.
 270-       N3=0.
 280-C
 290-C      INITIAL CONDITIONS U
 300-       U1(1)=0.
 310-       U2(1)=1.
 320-       U3(1)=10.
 330-       WRITE(6,1)I1,N1,I2,N2,I3,N3

 340-1      FORMAT(2X,F6.2,5X,F6.2,/,2X,F6.2,5X,F6.2,/,2X,F6.2,5X,F6.2)
 350-       WRITE(6,4)
 360-4      FORMAT(6X,2HU1,18X,2HU2,18X,2HU3,16X,1HI,16X,2HV1,
 370-      C20X,2HV2,20X,2HV3)
 380-C
 390-C      NORMALIZED MOMENTS OF INERTIA
 400-       A1=(I2-I3)/I1
 410-       B1=(I3-I1)/I2
 420-       C=(I1-I2)/I3
 430-C
 440-C      NORMALIZED TORQUES
 450-       M1=N1/I1

 460-       M2=N2/I2
 470-       M3=N3/I3
 480-C
 490-C      INITIAL VELOCITIES
 500-       V1(1)=0.
 510-       V2(1)=0.
 520-       V3(1)=0.
 530-C
 540-C      MASSES
 550-       MS1=100.
 560-       MS2=110.
 570-       MS3=120.
```

```
580-C
590-C      FORCES
600-       FRC1=10.
610-       FRC2=20.
620-       FRC3=40.
630-       ACL1=FRC1/MS1
640-       ACL2=FRC2/MS2
650-       ACL3=FRC3/MS3
660-       LIP=500.
670-       DT=.01
680-C      END INITIALIZATION
690-       DO 100 I=1,300,1

700-       TIME=TIME+DT
710-       T1=DT/3.
720-       T2=2.*DT/3.
730-       T5=.01
740-C      RIGHT HAND SIDES
750-       F1=M1+A1*U2(1)*U3(1)
760-       G1=M2+B1*U1(1)*U3(1)
770-       H1=M3+C*U1(1)*U2(1)
780-       X1=ACL1+U2(1)*U3(1)-U3(1)*U2(1)
790-       Y1=ACL2+U3(1)*U1(1)-U1(1)*U3(1)
800-       Z1=ACL3+U1(1)*U2(1)-U2(1)*U1(1)
810-C      COEFFICIENT MATRIX A

820-       A(1,1)=1.
830-       A(1,2)=2.*T1
840-       A(2,1)=1.
850-       A(2,2)=2.*T2
860-C       FORCING FUNCTION MATRICES
870-       B(1,1)=F1
880-       B(2,1)=F1
890-       B(1,2)=G1
900-       B(2,2)=G1
910-       B(1,3)=H1
920-       B(2,3)=H1
930-       B(1,4)=X1

940-       B(2,4)=X1
950-       B(1,5)=Y1
960-       B(2,5)=Y1
970-       B(1,6)=Z1
980-       B(2,6)=Z1
990-C      MATRIX SUBROUTINE PARAMETERS
1000-30    CONTINUE
1010-      M=6
1020-      N=2
1030-      IA=2
1040-      IDGT=5
1050-      CALL LEQT1F(A,M,N,IA,B,IDGT,USA,IER)
```

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```
1060-C     ASSEMBLE U S FOR INTERVAL
1070-      U1(2)=U1(1)+B(1,1)*T1+B(2,1)*T1**2
1080-      U1(3)=U1(1)+B(1,1)*T2+B(2,1)*T2**2
1090-      U2(2)=U2(1)+B(1,2)*T1+B(2,2)*T1**2
1100-      U2(3)=U2(1)+B(1,2)*T2+B(2,2)*T2**2
1110-      U3(2)=U3(1)+B(1,3)*T1+B(2,3)*T1**2
1120-      U3(3)=U3(1)+B(1,3)*T2+B(2,3)*T2**2
1130-      V1(2)=V1(1)+B(1,4)*T1+B(2,4)*T1**2
1140-      V1(3)=V1(1)+B(1,4)*T2+B(2,4)*T2**2
1150-      V2(2)=V2(1)+B(1,5)*T1+B(2,5)*T1**2
1160-      V2(3)=V2(1)+B(1,5)*T2+B(2,5)*T2**2
1170-      V3(2)=V3(1)+B(1,6)*T1+B(2,6)*T1**2

1180-      V3(3)=V3(1)+B(1,6)*T2+B(2,6)*T2**2
1190-      F2=M1+A1*U2(2)*U3(2)
1200-      F3=M1+A1*U2(3)*U3(3)
1210-      G2=M2+B1*U1(2)*U3(2)
1220-      G3=M2+B1*U1(3)*U3(3)
1230-      H2=M3+C*U1(2)*U2(2)
1240-      H3=M3+C*U1(3)*U2(3)
1250-      X2=ACL1+V2(2)*U3(2)-V3(2)*U2(2)
1260-      X3=ACL2+V2(3)*U3(3)-V3(3)*U2(3)
1270-      Y2=ACL2+V3(2)*U1(2)-V1(2)*U3(2)
1280-      Y3=ACL2+V3(3)*U1(3)-V1(3)*U3(3)
1290-      Z2=ACL3+V1(2)*U2(2)-V2(2)*U1(2)

1300-      Z3=ACL3+V1(3)*U2(3)-V2(3)*U1(3)
1310-C     NOW REDO INTERVAL
1320-      A(1,1)=1.
1330-      A(1,2)=2.*T1
1340-      A(2,1)=1.
1350-      A(2,2)=2.*T2
1360-      B(1,1)=F2
1370-      B(2,1)=F3
1380-      B(1,2)=G2
1390-      B(2,2)=G3
1400-      B(1,3)=H2
1410-      B(2,3)=H3

1420-      B(1,4)=X2
1430-      B(2,4)=X3
1440-      B(1,5)=Y2
1450-      B(2,5)=Y3
1460-      B(1,6)=Z2
1470-      B(2,6)=Z3
1480-      CALL LEQT1F(A,M,N,IA,B,IDGT,USA,IER)
1490-C     ASSEMBLE NEW U S FOR INTERVAL
1500-      U1(4)=U1(1)+B(1,1)*T5+B(2,1)*T5**2
1510-      U2(4)=U2(1)+B(1,2)*T5+B(2,2)*T5**2
1520-      U3(4)=U3(1)+B(1,3)*T5+B(2,3)*T5**2
1530-      V1(4)=V1(1)+B(1,4)*T5+B(2,4)*T5**2
```

```
1540-        U2(4)=U2(1)+B(1,5)*T5+B(2,5)*T5**2
1550-        V3(4)=U3(1)+B(1,6)*T5+B(2,6)*T5**2
1560-        WRITE(6,5)U1(4),U2(4),U3(4),I,IER,U1(4),U2(4),U3(4)
1570-5       FORMAT(2X,F10.4,10X,F10.4,10X,F10.4,5X,I3,3X,I2,9X,F10.4,
1580-       C10X,F10.4,10X,F10.4)
1590-        WRITE(5)TIME,U1(4),U2(4),U3(4),U1(4),U2(4),U3(4)
1600-        U1(1)=U1(4)
1610-        U2(1)=U2(4)
1620-        U3(1)=U3(4)
1630-        U1(1)=U1(4)
1640-        U2(1)=U2(4)
1650-        U3(1)=U3(4)

1660-        DT=DT+.01
1670-100     CONTINUE
1680-        END
```

```
L,ALL
100-        PROGRAM FEM1(INPUT,OUTPUT,TAPE5,TAPE6)
110-        DIMENSION A(4,4),B(4,5),UBA(24),T(5),F(5),G(5),H(5),
120-       CX(5),Y(5),Z(5)
130-        REAL I1,I2,I3,M1,M2,M3,M1,M2,M3,MS1,MS2,MS3
140-        REAL LIP
150-        DIMENSION U1(6),U2(6),U3(6),V1(6),V2(6),V3(6)
160-        TIME=0.
170-C
180-C       COLLOCATION PROGRAM
190-C       FOUR TERM APPROXIMATION
200-C       MOMENTS OF INERTIA
210-        I1=40.

220-        I2=40.
230-        I3=20.
240-C
250-C       TORQUES
260-        N1=0.
270-        N2=0.
280-        N3=0.
290-C
300-C INITIAL CONDITIONS W
310-        U1(1)=0.
320-        U2(1)=1.
330-        U3(1)=10.

340-        WRITE(6,1)I1,N1,I2,N2,I3,N3
350-1       FORMAT(2X,F6.2,5X,F6.2,/,2X,F6.2,5X,F6.2,/,2X,F6.2,5X,F6.2)
360-        WRITE(6,4)
370-4       FORMAT(6X,2HW1,18X,2HW2,18X,2HW3,16X,1HI,16X,2HW1,
380-       C20X,2HV2,20X,2HV3)
390-C
400-C       NORMALIZED MOMENTS OF INERTIA
410-        A1=(I2-I3)/I1
420-        B1=(I3-I1)/I2
430-        C=(I1-I2)/I3
440-C
450-C       NORMALIZED TORQUES

460-        M1=N1/I1
470-        M2=N2/I2
480-        M3=N3/I3
490-C
500-C       INITIAL VELOCITIES
510-        V1(1)=0.
520-        V2(1)=0.
530-        V3(1)=0.
540-C
550-C       MASSES
560-        MS1=100.
570-        MS2=110.
```

AFWAL-TR-81-1245

```
690-       M63=180.
695-C
696-C     FORCES
610-       FRC1=10.
620-       FRC2=20.
630-       FRC3=40.
640-       ACL1=FRC1/M51
660-       ACL2=FRC2/M62
660-       ACL3=FRC3/M63
670-       LIP=500.
680-       DT=.01
690-C     END INITIALIZATION

700-       DO 100 I=1,300,1
710-       TIME=TIME+DT
720-       T1=DT*.2
730-       T2=DT*.4
740-       T3=DT*.6
750-       T4=DT*.8
760-       T5=.01
770-C     RIGHT HAND SIDES
780-       F1=M1+A1*U2(1)*U3(1)
790-       G1=M2+B1*U1(1)*U3(1)
800-       H1=M3+C*U1(1)*U2(1)
810-       X1=ACL1+U2(1)*U3(1)-U3(1)*U2(1)

820-       Y1=ACL2+U3(1)*U1(1)-U1(1)*U3(1)
830-       Z1=ACL3+U1(1)*U2(1)-U2(1)*U1(1)
840-C     COEFFICIENT MATRIX A
850-       A(1,1)=1.
860-       A(1,2)=2.*T1
870-       A(1,3)=3.*T1*T2
880-       A(1,4)=4.*T1*T3
890-       A(2,1)=1.
900-       A(2,2)=2.*T2
910-       A(2,3)=3.*T2*T2
920-       A(2,4)=4.*T2*T3
930-       A(3,1)=1.

940-       A(3,2)=2.*T3
950-       A(3,3)=3.*T3*T2
960-       A(3,4)=4.*T3*T3
970-       A(4,1)=1.
980-       A(4,2)=2.*T4
990-       A(4,3)=3.*T4*T2
1000-      A(4,4)=4.*T4*T3
1010-C      FORCING FUNCTION MATRICES
1020-      B(1,1)=F1
1030-      B(2,1)=F1
1040-      B(3,1)=F1
1060-      B(4,1)=F1
```

```
1060-      B(1,2)=G1
1070-      B(2,2)=G1
1080-      B(3,2)=G1
1090-      B(4,2)=G1
1100-      B(1,3)=H1
1110-      B(2,3)=H1
1120-      B(3,3)=H1
1130-      B(4,3)=H1
1140-      B(1,4)=X1
1150-      B(2,4)=X1
1160-      B(3,4)=X1
1170-      B(4,4)=X1

1180-      B(1,5)=Y1
1190-      B(2,5)=Y1
1200-      B(3,5)=Y1
1210-      B(4,5)=Y1
1220-      B(1,6)=Z1
1230-      B(2,6)=Z1
1240-      B(3,6)=Z1
1250-      B(4,6)=Z1
1260-C     MATRIX SUBROUTINE PARAMETERS
1270-30    CONTINUE
1280-      M=6
1290-      N=4

1300-      IA=4
1310-      IDGT=5
1320-      CALL LEQT1F(A,M,N,IA,B,IDGT,WSA,IER)
1330-C     ASSEMBLE U S FOR INTERVAL
1340-      DO 400 L=2,5,1
1350-      T(2)=T1
1360-      T(3)=T2
1370-      T(4)=T3
1380-      T(5)=T4
1390-      U1(L)=U1(1)+B(1,1)*T(L)+B(2,1)*T(L)**2+B(3,1)*T(L)**3+
1400-     C B(4,1)*T(L)**4
1410-      U2(L)=U2(1)+B(1,2)*T(L)+B(2,2)*T(L)**2+B(3,2)*T(L)**3+

1420-     C B(4,2)*T(L)**4
1430-      U3(L)=U3(1)+B(1,3)*T(L)+B(2,3)*T(L)**2+B(3,3)*T(L)**3+
1440-     C B(4,3)*T(L)**4
1450-      V1(L)=V1(1)+B(1,4)*T(L)+B(2,4)*T(L)**2+B(3,4)*T(L)**3+
1460-     C B(4,4)*T(L)**4
1470-      V2(L)=V2(1)+B(1,5)*T(L)+B(2,5)*T(L)**2+B(3,5)*T(L)**3+
1480-     C B(4,5)*T(L)**4
1490-      V3(L)=V3(1)+B(1,6)*T(L)+B(2,6)*T(L)**2+B(3,6)*T(L)**3+
1500-     C B(4,6)*T(L)**4
1510-      F(L)=M1+A1*U2(L)*U3(L)
1520-      G(L)=M2+B1*U1(L)*U3(L)
1530-      H(L)=M3+C1*U1(L)*U2(L)
```

```
1540-        X(L)=ACL1+U2(L)*U3(L)-V3(L)*U2(L)
1550-        Y(L)=ACL2+V3(L)*U1(L)-V1(L)*U3(L)
1560-        Z(L)=ACL3+V1(L)*U2(L)-U2(L)*U1(L)
1570-400    CONTINUE
1580-C      NOW REDO INTERVAL
1590-       A(1,1)=1.
1600-       A(1,2)=2.*T1
1610-       A(1,3)=3.*T1**2
1620-       A(1,4)=4.*T1**3
1630-       A(2,1)=1.
1640-       A(2,2)=2.*T2
1650-       A(2,3)=3.*T2**2

1660-       A(2,4)=4.*T2**3
1670-       A(3,1)=1.
1680-       A(3,2)=2.*T3
1690-       A(3,3)=3.*T3**2
1700-       A(3,4)=4.*T3**3
1710-       A(4,1)=1.
1720-       A(4,2)=2.*T4
1730-       A(4,3)=3.*T4**2
1740-       A(4,4)=4.*T4**3
1750-C
1760-C
1770-       B(1,1)=F(2)

1780-       B(2,1)=F(3)
1790-       B(3,1)=F(4)
1800-       B(4,1)=F(5)
1810-       B(1,2)=G(2)
1820-       B(2,2)=G(3)
1830-       B(3,2)=G(4)
1840-       B(4,2)=G(5)
1850-       B(1,3)=H(2)
1860-       B(2,3)=H(3)
1870-       B(3,3)=H(4)
1880-       B(4,3)=H(5)
1890-       B(1,4)=X(2)

1900-       B(2,4)=X(3)
1910-       B(3,4)=X(4)
1920-       B(4,4)=X(5)
1930-       B(1,5)=Y(2)
1940-       B(2,5)=Y(3)
1950-       B(3,5)=Y(4)
1960-       B(4,5)=Y(5)
1970-       B(1,6)=Z(2)
1980-       B(2,6)=Z(3)
1990-       B(3,6)=Z(4)
2000-       B(4,6)=Z(5)
2010-       CALL LEQT1F(A,M,N,IA,B,IDGT,WBA,IER)
```

```
2020-C        ASSEMBLE NEW U S FOR INTERVAL
2030-         U1(6)=U1(1)+B(1,1)ST5+B(2,1)ST5**2+B(3,1)ST5**3+
2040-        CB(4,1)ST5**4
2050-         U2(6)=U2(1)+B(1,2)ST5+B(2,2)ST5**2+B(3,2)ST5**3+
2060-        CB(4,2)ST5**4
2070-         U3(6)=U3(1)+B(1,3)ST5+B(2,3)ST5**2+B(3,3)ST5**3+
2080-        CB(4,3)ST5**4
2090-         V1(6)=V1(1)+B(1,4)ST5+B(2,4)ST5**2+B(3,4)ST5**3+
2100-        CB(4,4)ST5**4
2110-         V2(6)=V2(1)+B(1,5)ST5+B(2,5)ST5**2+B(3,5)ST5**3+
2120-        CB(4,5)ST5**4
2130-         V3(6)=V3(1)+B(1,6)ST5+B(2,6)ST5**2+B(3,6)ST5**3+

2140-        CB(4,6)ST5**4
2150-         WRITE(6,5)U1(6),U2(6),U3(6),I,IER,V1(6),V2(6),V3(6)
2160-5       FORMAT(2X,F10.4,10X,F10.4,10X,F10.4,5X,I3,3X,I2,9X,F10.4,
2170-        C10X,F10.4,10X,F10.4)
2180-        WRITE(5)TIME,U1(6),U2(6),U3(6),V1(6),V2(6),V3(6)
2190-        U1(1)=U1(6)
2200-        U2(1)=U2(6)
2210-        U3(1)=U3(6)
2220-        V1(1)=V1(6)
2230-        V2(1)=V2(6)
2240-        V3(1)=V3(6)
2250-        DT=DT+.01

2260-100    CONTINUE
2270-       END
```

```
L<ALL
.ALL
  100=     PROGRAM RK(INPUT,OUTPUT,TAPE5,TAPE6)
  110=     REAL I1,I2,I3,N1,N2,N3,A1,A2,A3,MS1,MS2,MS3
  120=     REAL KF(4),KG(4),KH(4),KX(4),KY(4),KZ(4)
  130=     REAL LIP
  140=     DIMENSION U1(4),U2(4),U3(4),V1(4),V2(4),V3(4)
  150=C    RUNGE KUTTA PROGRAM
  160=C
  170=     TIME=0.
  180=C    MOMENTS OF INERTIA
  190=     I1=40.
  200=     I2=30.
  210=     I3=20.

  220=C
  230=C    TORQUES
  240=     N1=100.
  250=     N2=200.
  260=     N3=150.
  270=     WRITE(6,1)I1,N1,I2,N2,I3,N3
  280=1     FORMAT(2X,F6.2,5X,F6.2,/,2X,F6.2,5X,F6.2,/,2X,F6.2,5X,F6.2)
  290=C
  300=C    INITIAL CONDITIONS U
  310=     U1(1)=0.
  320=     U2(1)=1.
  330=     U3(1)=1.

  340=C
  350=C    NORMALIZED MOMENTS OF INERTIA
  360=     A=(I2-I3)/I1
  370=     B=(I3-I1)/I2
  380=     C=(I1-I2)/I3
  390=C
  400=C    NORMALIZED TORQUES
  410=     A1=N1/I1
  420=     A2=N2/I2
  430=     A3=N3/I3
  440=C
  450=C    INITIAL VELOCITIES

  460=     V1(1)=0.
  470=     V2(1)=0.
  480=     V3(1)=0.
  490=C
  500=C    MASSES
  510=     MS1=100.
  520=     MS2=110.
  530=     MS3=120.
  540=C    FORCES
  550=     FRC1=10.
  560=     FRC3=40.
  570=     FRC2=20.
```

```
600-C        CALCULATE ACCELERATIONS
610*         ACL1=FRC1/AS1
620*         ACL2=FRC2/AS2
630*         ACL3=FRC3/AS3
640*         WRITE(6,7)
650-C
660-C        DT=.01
670-C
680*         UU1=U1(1)
690*         UU2=U2(1)
700*         UU3=U3(1)
710*         VV1=V1(1)

720*         VV2=V2(1)
730*         VV3=V3(1)
740-C
750-C        END INITIALIZATION
760-C
770*         DO 300 L=1,100,1
780*         TIME = TIME+DT
790*         DO 100 I=1,2,1
800*         J=I-1
810*         IF(I.EQ.1)J=1
820*         KF(I)=R1+A*U2(J)*U3(J)
830*         KG(I)=R2+B*U3(J)*U1(J)

840*         KH(I)=R3+C*U1(J)*U2(J)
850*         KX(I)=ACL1+U2(J)*U3(J)-U3(J)*U2(J)
860*         KY(I)=ACL2+U3(J)*U1(J)-U1(J)*U3(J)
870*         KZ(I)=ACL3+U1(J)*U2(J)-U2(J)*U1(J)
880*         U1(I)=U1(J)+.5*DT*KF(I)
890*         U2(I)=U2(J)+.5*DT*KG(I)
900*         U3(I)=U3(J)+.5*DT*KH(I)
910*         V1(I)=V1(J)+.5*DT*KX(I)
920*         V2(I)=V2(J)+.5*DT*KY(I)
930*         V3(I)=V3(J)+.5*DT*KZ(I)
940=100      CONTINUE
950*         I = 3

960*         J = 2
970*         KF(I)=R1+A*U2(J)*U3(J)
980*         KG(I)=R2+B*U3(J)*U1(J)
990*         KH(I)=R3+C*U1(J)*U2(J)
1000*        KX(I)=ACL1+U2(J)*U3(J)-U3(J)*U2(J)
1010*        KY(I)=ACL2+U3(J)*U1(J)-U1(J)*U3(J)
1020*        KZ(I)=ACL3+U1(J)*U2(J)-U2(J)*U1(J)
1030*        U1(I)=U1(J)+DT*KF(I)
1040*        U2(I)=U2(J)+DT*KG(I)
1050-C
1060-C       END DETERMINATION OF K'S.
1070-C
```

```
        CONTINUE
        FORMAT(1X,F10.4,10X,F10.4,10X,F10.4,10X,I3,5X,F10.4,10X,
       CF10.4,10X,F10.4)
        FORMAT(6X,SMU1,15X,SMU2,15X,SMU3,15X,1HL,10X,SMU1,
       C15X,SMU2,15X,SMU3)
        CONTINUE
        END
        RETURN
        STOP
        EOF
```

```
1000-        U3(I)=U3(J)+DT*KH(I)
1070-        U1(I)=U1(J)+DT*KX(I)
1080-        U2(I)=U2(J)+DT*KY(I)
1090-        U3(I)=U3(J)+DT*KZ(I)
1100-        I=4
1110-        J=3
1120-        KF(I)=R1+A*U2(J)*U3(J)
1130-        KG(I)=R2+B*U3(J)*U1(J)
1140-        KH(I)=R3+C*U1(J)*U2(J)
1150-        KX(I)=ACL1+U2(J)*U3(J)-U3(J)*U2(J)
1160-        KY(I)=ACL2+U3(J)*U1(J)-U1(J)*U3(J)
1170-        KZ(I)=ACL3+U1(J)*U2(J)-U2(J)*U1(J)

1180-        U1(I)=UU1+(DT/6.)*(KF(1)+2.*(KF(2)+KF(3))+KF(4))
1190-        U2(I)=UU2+(DT/6.)*(KG(1)+2.*(KG(2)+KG(3))+KG(4))
1200-        U3(I)=UU3+(DT/6.)*(KH(1)+2.*(KH(2)+KH(3))+KH(4))
1210-        U1(I)=UU1+(DT/6.)*(KX(1)+2.*(KX(2)+KX(3))+KX(4))
1220-        U2(I)=UU2+(DT/6.)*(KY(1)+2.*(KY(2)+KY(3))+KY(4))
1230-        U3(I)=UU3+(DT/6.)*(KZ(1)+2.*(KZ(2)+KZ(3))+KZ(4))
1240-        WRITE(6,5)U1(I),U2(I),U3(I),L,U1(I),U2(I),U3(I)
1250-        WRITE(5)TIME,U1(I),U2(I),U3(I),U1(I),U2(I),U3(I)
1260-CC       LIPSCHITZ CONDITIONS
1270-        FL2=ABS(U1(I)-UU1)
1280-        GL2=ABS(U2(I)-UU2)
1290-        HL2=ABS(U3(I)-UU3)

1300-        XL2=ABS(U1(I)-UU1)
1310-        YL2=ABS(U2(I)-UU2)
1320-        ZL2=ABS(U3(I)-UU3)
1330-        FL1=ABS(KF(4)-KF(1))
1340-        GL1=ABS(KG(4)-KG(1))
1350-        HL1=ABS(KH(4)-KH(1))
1360-        XL1=ABS(KX(4)-KX(1))
1370-        YL1=ABS(KY(4)-KY(1))
1380-        ZL1=ABS(KZ(4)-KZ(1))
1390-        LIP=500.
1400-C
1410-        IF(XL1.GT.XL2*LIP)GO TO 8

1420-        IF(YL1.GT.YL2*LIP)GO TO 8
1430-        IF(ZL1.GT.ZL2*LIP)GO TO 8
1440-        IF(FL1.GT.FL2*LIP)GO TO 8
1450-        IF(GL 1.GT.GL2*LIP)GO TO 8
1460-        IF(HL1.GT.HL2*LIP)GO TO 8
1470-C
1480-        UU1 = U1(I)
1490-        UU2 = U2(I)
1500-        UU3 = U3(I)
1510-        UU1=U1(I)
1520-        UU2=U2(I)
1530-        UU3=U3(I)
```